Bayesian Risk Assessment Technique for Economic Stress-Strength Models

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Abstract:

This paper explores two areas of risk assessment modelling in economics and business: the Stress-Strength model and Bayesian techniques. The model assumes that the probability of stress exceeding strength is a measure of risk. The interpretation of stress and strength largely depends on the particular event or phenomenon being modelled. The use of the Stress-Strength model is demonstrated through the Gaussian assumption of probability distributions for random model parameters, particularly in assessing the risk of not achieving a required margin value. The concept of the capability function, representing the difference between strength and stress, is introduced in the modelling process. The probability distribution for the capability function is initially determined based on the Gaussian distribution of the random variables used in the model, allowing for evaluating the risk metric. The Bayesian approach is then applied to generalise the problem statement when dealing with unknown parameters of probability distributions for the Stress and Strength models. The uncertainty of these parameters is modelled through uniform probability distributions, and equations for calculating prior and posterior estimates are consistently obtained. Since multidimensional integrals are involved in these calculations, and solutions cannot be obtained in closed analytical form, Monte Carlo simulation is used to solve this computation problem.

Keywords: stress-strength model; capability function; Gaussian; Bayesian.

JEL Classification: C11, G32; E21; E17.

Introduction

Many approaches and definitions arise regarding which business should be recognised as effective and which should not. The first of the key factors determining a business's effectiveness is its financial performance. Another factor is customer satisfaction. A business that can meet the needs and expectations of its customers is likely to have a loyal customer base and generate repeat business. Employee productivity is also a crucial factor in determining a business's effectiveness. Finally, market share is another factor that determines the effectiveness of a business that can capture a significant portion of the market share in its industry and is likely to be successful and profitable. All these factors can be combined into one: a business can be defined as effective if it is able to achieve its strategic goals, which can be defined in terms of financial categories, customer satisfaction, market share, etc. The desire to deeply understand the reasons for not achieving the company's strategic goals involves building adequate models that effectively describe the conditions in which these goals occur.

It must be understood that there is always a risk of failure to achieve the set strategic one. To assess this risk before or during the execution of a strategic plan, a model must be developed. The purpose of the model is to quantify the risk to a decision-maker. The final decision will depend on his or her risk appetite. One way or another, the model must be constructed and consider most factors accompanying achieving the strategic goals.

A special kind of modelling is considered in this paper, using a so-called Stress-Strength Model (SSM). The model describes the solution to any economic problem, which reduces to a joint analysis of (i) the capabilities of the system (performance, power, and so on) and (ii) the impact on the system from the external and internal factors for the system.

As a typical example, consider a case when needed to determine whether it would be provided with enough funds to fulfil an investment project, which must serve as a necessary condition for achieving a strategic goal. On the one hand, we are assessing the demand for funds for the project to be successful, and on the other hand, we need to model the availability of such funds. For this case, we define funds availability as a strength, whereas demand can qualify as a Stress. In dead, the company evaluates its ability to allocate funds for the implementation, for example, of a marketing program, and this qualifies as its Strength. At the same time, an assessment is made of the project's demand for financing to achieve the strategic goal. This can be interpreted as Stress for the business since the company could not provide the necessary financing. The risk that the required financing will not be able to be provided casts doubt on the possibility of implementing the project and achieving the strategic goal.

1. Literature Review

The Stress and Strength models have widespread usage across various fields of knowledge and applications, (Johnson, Kotz and Balakrishnan,1995). There exists a valid generalization of the Stress-Strength model (Kotz, Lumelskii and Penski, 2003). It is important to note that most applications of the SSM analyse the reliability of mechanical structures by comparing the load on the object with its tensile strength. If the load exceeds the strength, the object collapses; otherwise, it is considered workable, achieving the goal of the mechanical structure.

Despite the apparent usefulness of interpreting the ratio in the context of Stress-Strength analysis for economic and business applications, the SSM is not given the attention it deserves when assessing economic risks. Within the framework of economics and business, the model suggests that stress, which can be defined as any external factor that creates a demand on an individual or organization, can be managed by building up strength. Strength refers to the internal resources and capabilities that enable an individual or organization to cope with stress. An objective description of using models in the spirit of SSM can be found in Vose (2008).

It is worth dwelling on another feature of solving the problem under consideration. The fact is that the assessment of the risk of achieving a strategic goal is based on the model that is used for evaluation. As it's well-known, there are no perfect models at all. Model imperfections can generally arise due to various factors, including simplifying assumptions made during model development, inaccuracies in measurement data used to develop the model, and limitations in the computational techniques used to solve the model equations. In some cases, model imperfections may be negligible and have little impact on the accuracy of the model predictions and following risk assessment. However, in other cases, model imperfections can significantly affect conclusions and decision-making (Savchuk, 1995).

A model is always built based on some assumptions and fundamentally cannot consider all the factors involved. That is, it contains an error. In other words, here we are faced with two kinds of uncertainty. The model's parameters create the first uncertainty - we will never be able to unambiguously quantify all the factors acting on the process of achieving the goal. Therefore, we should simulate this uncertainty using random variables or fuzzy numbers. At the same time, we do not have confidence in the flawlessness of the model itself, and we must also take this uncertainty into account and use it to assess the risk of achieving a strategic goal.

How to cope with the uncertainty of the model used and the variables used in it. This is where the Bayesian approach comes to the rescue. Firstly, we must stress that "Bayesian" is used differently in different disciplines. In statistics and computer science, for instance, anything that updates a prior to a posterior based on evidence is called "Bayesian". In economic application, by contrast, the term "Bayesian" refers to a more demanding methodological position, according to which anything and everything unknown should be modelled explicitly in a state-space model and subject to a prior probability. This position is sometimes referred to as "Bayesianism" (Jeffreys, 1961, DeGroot, 1975, Lee, 2004, Savchuk and Tsokos, 2014).

We will consider that the Bayesian approach to risk assessment and decision-making is a statistical method that involves updating prior information with new data to form posterior information. In more detail, the Bayesian approach consists of specifying a prior distribution for the parameters of interest, which reflects the decision-makers' prior beliefs about the values of those parameters. This prior distribution is then updated with new data using Bayes' rule to obtain the posterior distribution, which reflects the researcher's updated beliefs about the values of the parameters. The advantage of the Bayesian approach is that it allows decision-makers to incorporate prior knowledge into their modelling process, leading to more accurate estimates and predictions. As a rule, the choice of the prior distribution is subjective and may influence the resulting posterior distribution. As new data become available, subjective assessments lose their influence. However, the data is usually insufficient for the necessary assessments to make reliable decisions. And then, their role is performed by the prior subjective assumptions.

The main purpose of the paper is to solve the problem of risk estimation based on the Stress-Strength Model (SSM), taking care of the uncertainty of data used for decision-making. The solution is based on a Bayesian method, which is fundamental for decision-making while estimating a risk measure. All considerations in the paper use a Gaussian distribution for random variables of the SSM. This assumption is made for the sake of simplicity in obtaining final results. It should also be mentioned that this instance is widespread in the actual practice risk estimation.

2. Stress-Strength Model Risk Assessment

The SSM's concept is a basis for estimating risk metrics for any economic phenomenon or event. The model is based on combining strength, say Q, and stress, say S. An unfavourable event occurs if the stress exceeds the strength. For example, if the fund demand for a project exceeds available funds, as mentioned before. A similar consideration can be used in the enterprise's current activity. In this case, the role of strength G is played by an enterprise's total revenue over a given period, whereas the stress S is modelled by its total expenses. An unfavourable event occurs when the total expenses exceed the revenue. The probability of such a situation can be assigned as a risk of current business activity.

First, we consider the SSM model in the latter case and demonstrate the risk assessment technique. We assume the model's variables are Gaussian random with known parameters of probability density function (pdf). In the next section, we will generalise the risk assessment procedure in the Bayesian sense, assuming the uncertainty of the pdf's parameters.

Here, we consider the case when both the strength and stress are assumed to be time-invariant. The essential feature of the model concept is that the strength as well the stress are functions of some economic variables which form a vector $X = (X_1, X_2, ..., X_n)$. This vector is a formalised representation of economic indicators (product price, purchase price, etc.) included in the budget. The company's successful activity consists of the fact that the stress does not exceed the strength. Since the economic variables of the budget are random, the model involves using a margin of safety M. One can understand the introduction of such a margin by considering the instance where the Stress is total expenses, and the Strength is the total revenue for the given period. Two cases are possible. The risk of the company's current activities is that the company should not be unprofitable. A more stringent requirement is that the company's current activities must provide it with a given profit value. In the first case, M = 0. In the second one, the M > 0, and its number meets the requirements of the owner or CEO.

Thus, the company risk can be presented by the following expression:

$$R = Pr.\{S(X) + M > G(X)\} \tag{2.1}$$

Let's now dwell on the case when SSM presents the continuous budgeting process and the goal of the enterprise's current activity is to provide a required value of operating profit, that is, earnings before interest and taxes (EBIT). Variable μ plays the role of the required EBIT.

To simplify further reasoning, we introduce the denotation of the capability function, which is a difference between strength added by the margin of safety and stress:

$$U(X) = G(X) - S(X) - M \tag{2.2}$$

For this assumption eq. (2.1) can be presented as follows:

$$R = Pr.\{U(X) = G(X) - S(X) - M < 0\} \tag{2.3}$$

The probability estimation problem (2.3) is easily solved for Gaussian stress and strength. In general, estimating this probability does not seem simple since the functions $S(X), G(X)$, and hence the $U(X)$, are generally nonlinear. The assumption of Gaussian variables X also does not help since a nonlinear transformation of Gaussian variables does not lead to a Gaussian value. Therefore, only an approximate solution is possible. To obtain an approximate solution, we will make the following assumptions:

- the variable vector $X = (X_1, X_2, \dots, X_n)$ obeys the multidimensional Gaussian distribution,
- function (2.2) is roughly represented as the linear part of the Taylor approximation, which is:

$$U(X) \cong U(m_X) + \sum_{i=1}^{n} \frac{\partial U(m_X)}{\partial m_i}(X_i - m_i), \qquad (2.4)$$

where: $m_X = (m_1, m_2, \dots, m_n)$ is a vector of mean values of $X = (X_1, X_2, \dots, X_n)$, that is $m_i = E(X_i)$.

Using the capability function approximation should not be alarming since, ultimately, we are estimating the magnitude of the risk with this approximation. It And this estimate is not an exact numerical value. A minor difference in estimates, such as 0.24 and 0.27, will be interpreted similarly by a decision-maker.

Returning to the problem of assessing the risk of not obtaining a given EBIT value, consider the case when the company's product portfolio is represented by N items so that Strength (total revenue) and Stress (Total Costs) are presented as follows:

$$G = \sum_{k=1}^{N} p_k \cdot Q_k, \qquad S = \sum_{k=1}^{N} v_k \cdot Q_k + F,$$

where: p_k is the price for the k-th item, v_k is its per unit variable cost, Q_k denotes a projected sale volume of the k-th item, and F is the total fixed costs.

Then the final equation for the capability function can be presented as:

$$U = \sum_{k=1}^{N} (p_k - v_k) \cdot Q_k - F - \text{M} \qquad (2.5)$$

As we see from (2.5), there are three sets of variables:

- $Q_p = (Q_1, Q_2, \dots, Q_N)$,
- $m_p = (p_1, p_2, \dots, p_N)$, $m_v = (v, v_2, \dots, v_N)$, which are assumed to be Gaussian random and can be fully denoted by their mean values, standard deviations and correlation matrix.

We restrict the model by the case when the pairs of (p_k, v_k), (p_k, Q_k), (v_k, Q_k) , (k = 1, 2, ...N) are subject of correlation. In other words, we assume that correlations are subject to (i) the unit's selling price and its variable cost, (ii) the unit's selling price and its sale volume, (iii) the unit's variable cost and its sale volume. This assumption corresponds to the situation when goods on the market are not interchangeable and promoted in the market independently.

By substituting (2.5) into (2.4), determine the expected value of the capability function:

$$m_U \cong \sum_{k=1}^{N} \left(m_{(p_k} - m_{(v_k)} \right) \cdot m_{Q_k} - m_F - \text{M}. \qquad (2.6)$$

This means that we approximated the expected value of the function of random variables by the function of the mean values of these random variables. Of course, we get an approximate value of the expected value and the error of such an approximation is less the smaller the standard deviations of random variables.

Similarly, we can determine the capability function variance. First, we present its approximate value for the general case of (2.4), and then simplify it for the case (2.5). The first step gives us the following general expression:

$$\sigma_U^2 = Var(U) = E[(U - m_U)^2] = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial U(m_X)}{\partial m_i} \cdot \frac{\partial U(m_X)}{\partial m_j} \cdot \rho_{ij} \cdot \sigma_i \cdot \sigma_j, \qquad (2.7)$$

where: ρ_{ij} $(i, j = 1, 2, \dots n)$ form a matrix of correlation coefficients $\mathbb{R}$.

Let's transfer (2.7) to the capability function (2.5). Assume that all the products in the company's portfolio are selling independently, which corresponds to the situation when we take into account the correlation between the pairs $(p_k, v_k), (p_k, Q_k), (v_k, Q_k), k = 1,2, ... N$. In this case correlation matrix can be presented as follows:

$$\mathbb{R} = \begin{pmatrix} \rho_{p_1 v_1} & \rho_{p_2 v_2} & \cdots & \rho_{p_N v_N} \\ \rho_{p_1 Q_1} & \rho_{p_2 Q_2} & \cdots & \rho_{p_N Q_N} \\ \rho_{v_1 Q_1} & \rho_{v_2 Q_2} & \cdots & \rho_{v_N Q_N} \end{pmatrix}.$$

Under these assumptions (2.7) can be rewritten into the following closed-formed equation:

$$\sigma_U^2 = Var(U) \cong \sum_{k=1}^{N} \left[m_{Q_k}^2 \sigma_{p_k}^2 + m_{Q_k}^2 \sigma_{v_k}^2 + \left(m_{p_k} - m_{v_k} \right)^2 \sigma_{Q_k}^2 \right] +$$

$$+2 \sum_{k=1}^{N} \left[m_{Q_k} \left(m_{p_k} - m_{v_k} \right) \left(\rho_{p_k Q_k} \sigma_{p_k} \sigma_{Q_k} - \rho_{v_k Q_k} \sigma_{v_k} \sigma_{Q_k} \right) - m_{Q_k}^2 \rho_{p_k v_k} \sigma_{p_k} \sigma_{v_k} \right] + \sigma_F^2 \qquad . (2.8)$$

Since we found the expected value m_U and standard deviation σ_U of the capability function U under the assumption of its Gaussian distribution, we can compute the risk metric (2.3):

$$R = Pr.\{U(X) < 0\} = \Phi\left(\frac{m_U}{\sigma_U} \right), \qquad (2.9)$$

where: $\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{x^2}{2}}$.

Consider a collection of three products characterized by the following economic parameters:

Table 2.1. Products' portfolio (all monetary numbers are in $1000)

Product #1	Mean Value	St. Dev.
Price per Unit	10,00	1,00
Variable Cost per Unit	5,50	0,60
Sale Volume	30,00	3,00

Product #2	Mean Value	St. Dev.
Price per Unit	8,50	0,70
Variable Cost per Unit	4,60	0,44
Sale Volume	20,00	2,40

Product #3	Mean Value	St. Dev.
Price per Unit	12,00	1,10
Variable Cost per Unit	7,50	0,65
Sale Volume	15,00	1,20

Table 2.2. Correlation matrix

Pair/Product	product #1		product #2		product #3	
Price/VC		0,35		0,40		0,60
Price/Volume	-	0,40	-	0,45	-	0,30
VC/Volume	-	0,30	-	0,35	-	0,30

The Mean Value and Standard Deviation of the portfolio's fixed costs are estimated to be $160K and $12K, respectively. Using (2.6) and (2.8), we can calculate the Mean Value and Standard Deviation of the Capability Function U: $m_U = \$75.50K$, $\sigma_U = \$37.12K$. If the management requires EBIT value M $= \$45K$, using (2.9), we can finally find the risk estimate as *0.21*.

The risk assessments were made possible due to the known parameters listed in Table 2.1 and Table 2.2 for the product portfolio. However, in real-life situations, obtaining such precise information is often impossible. Decision makers must rely on educated assumptions about parameters' expected values and deviations, such as sales volumes and prices. As more information becomes available, these assumptions can be refined using the Bayesian approach to convert prior information into posterior information. In the following section, we will provide a brief description of the Bayesian approach and demonstrate how it is used to obtain risk assessments through the application of SSM.

3. General Bayesian Methodology and Technique

From a practical standpoint, the Bayesian approach combines the following three statements.

Statement 1. The parameter of the system or model under study is assumed to be uncertain, and this uncertainty is modelled by means of a random variable. Before observation, the prior probability distribution of the parameter is assumed to be known. It should be noted that here we are now considering secondary randomness. Primary randomness models a primary random variable that describes a process or model. In the considered SSM, the role of the main random variable is played by the capability function. At the same time, secondary randomness describes the uncertainty of the parameters of its probability distribution.

Statement 2. A posterior distribution is obtained by combining the prior distribution of the parameter (describing secondary randomness) with the results of the observation of the main random variable. These observations are modelled by using the so-called likelihood function. This combining is made using the Bayes' rule.

Statement 3. A final decision making is made by maximising the expected utility or minimising the losses associated with the application of this rule. In the most practical application, the squared-error loss function is used, which leads to the estimation of the parameters or any of its functions as a posterior mean value.

Unlike classical decision theory, which assumes that the parameter of a probability distribution for the primary variable (the capability function in this case) is non-random, Bayesian theory assumes that the parameter is random. This randomness can be interpreted in a frequentist sense, where the parameter's value is generated by a stable and fundamental random mechanism whose properties are either known or can be obtained by analysing corresponding observations. For example, the parameter could be the price of a batch of a specific resource consumed continuously in the production process. In this case, observations of previous batches make it possible to estimate the prior distribution, provided that the initial technological process is sufficiently stable. For this situation, a frequency interpretation of the probability is appropriate, though it is as desirable as rare.

The Bayesian theory's most challenging question is estimating subjective probabilities and quantifying subjective experiences. In the Bayesian methodology, the interpretation of judgments is always probabilistic and can be represented by means of:

- a frequency (objective) interpretation of probability, which is extremely rare since it requires many past experiences.
- rational degrees of certainty are mainly reduced to the mathematical expression of the absence of a priori knowledge.
- subjective beliefs refer to the researcher's attitude towards the phenomenon or system under study.

The areas of application of these methods practically do not intersect. In the first case, in the presence of many past observations, both rationalistic and subjectivist positions, the levels of belief inevitably coincide with relative frequencies. In the complete absence of knowledge, subjective levels of belief must coincide with rational ones, i.e., with the need to accept a uniform prior distribution. In all other situations, and they are the exclusive majority, subjective levels of belief are a unique way of presenting prior information.

The methodical basis of the process of transition from prior information, formalised in the form of a prior distribution, to a posterior one by adding observation is Bayes' rule. This process can be represented as a sequential accumulation of information. At the initial stages of studying a phenomenon, a decision maker with certain qualifications and experience of past similar works has some idea of the properties of the object under studying. This view, in addition to non-formalized experience, includes empirical data obtained earlier with similar studies. During observation, new information appears in the form of a set of data that changes the representation (probabilistic judgment) of the properties of the object. Thus, at the same time, there is a gradual revision and reassessment of the prior presentation. Moreover, at each moment, we can give a complete description of the properties of the object, and this description will be exhaustive in the sense that we have used all the available information for it. This process does not stop – it continues with each new observed result.

Bayesian decision-making scheme consists of four components (Savchuk and Tsokos, 2014).

1. *The model, represented by the probability space $(\Omega, \mathcal{L}, P)$.* Here $\Omega = \{\boldsymbol{X}\}$ is the set of all possible data in some domain Π, which reflects random variables that contribute to profit. $\boldsymbol{X}$ is the data of a random experiment, thus on Ω it is determined some σ-algebra $\mathcal{L}$ of random events; $P \in \mathcal{B}$, where $\mathcal{B}$ is the family of probability measures on $(\Omega, \mathcal{L})$. In the traditional Bayesian approach, the probability measure P is defined by the representation of some parameter θ (vector or scalar), that is, $\mathcal{B} = \{P_\theta ; \theta \in \Theta\}$ is a parameterized family of probability measures.

2. *The probability space $(\Theta, \mathcal{E}, H)$ for the parameter θ which is assumed to be random. Here $\mathcal{E}$ is σ - algebra on (Θ, H) is a probability measure on $(\Theta, \mathcal{E})$. The measure H is called a prior probability measure of the parameter θ. The prior measure H belongs to some given family of probability measures $\mathcal{H}$.*
3. *The set of such possible decisions D that each element d from D is a measurable function on Ω. The set of decisions D contain all estimators of the parameter θ or some function $R(\theta)$ measurable on Ω.*
4. *The loss functions $L(\theta, d)$ or $L(R(\theta), d)$ determined on $\Theta \times D$. This loss function determines the losses caused by the replacement of the parameter θ by the decision element d.*

Let's denote probability density functions (pdf) $f(x|\theta)$ and $h(\theta)$, where $x \in \Omega, \theta \in \Theta$ and assume the joint density of the probability distribution for the random variables X and θ takes on the form: $g(x, \theta) = f(x|\theta)h(\theta)$. In accordance with the Bayes theorem, the conditional density for θ given $X = x$ is called the posterior pdf of the parameter θ and is written as:

$$\hbar(\theta|x) \sim h(\theta) \cdot f(x|\theta) \qquad (3.1)$$

taking into account the fact that the normalising factor of the pdf $\hbar(\theta|x)$ is found from the integral:

$$\beta = \left[\int h(\theta)f(x|\theta)d\theta \right]^{-1}. \qquad (3.2)$$

An extensive range of tasks for the Bayesian approach to risk assessment opens up economic and business applications. Since managers make many decisions based on subjective ideas and personal experience, it is often not economically feasible to perform expensive experiments that require diversifying resources and time. In this case, the manager needs a convenient and accurate methodology for assessing risk.

4. Bayesian Risk Estimations for Stress-Strength Model

This section discussed the classic solution to the problem involving risk assessment. We assumed that the parameters $\theta \in \Theta$ for pdf of the random variables that contribute to profit were unambiguously defined. However, in most real-life scenarios, we are unable to determine their exact values, which introduces uncertainty into the equations of Section 3. Instead, we can make some judgments based on our subjective initial information I_a regarding uncertain parameters. Then we can formalise them by a prior distribution $h(\theta|I_a)$. As we mentioned above, the essence of the Bayesian technique is in assembling a piece of prior information with observations of the main random variables that contribute to profit, denoted in Section 2 as X.

Let's $\omega = (X_1, X_2, ..., X_m)$ is a sample collected while observing X. Applying the Bayesian rule (3.1) and (3.2) we will use the so-called likelihood function $l(\theta|\omega)$, which represents the probability density function of the observations presented as a function of the parameter θ. For such an interpretation of Bayes' rule, we replace $f(x|\theta)$ by $l(\theta|\omega)$ in (3.1). In this form, the Bayes formula is often used to solve applied problems. Finally, we arrive at the following expression of the Bayes' rule:

$$\hbar(\theta|I_a, \omega) = \frac{h(\theta|I_a) \cdot l(\theta|\omega)}{\int h(\theta|I_a) \cdot l(\theta|\omega)d\theta}, \qquad (4.1)$$

where: $\hbar(\theta|I_a, \omega)$ is conditional with respect to the initial information I_a, and observations ω.

The diagram for reconsidering probability, presented in Figure 4.1, illustrates the transformation from prior information to posterior information

Figure. 4.1. Bayesian technique
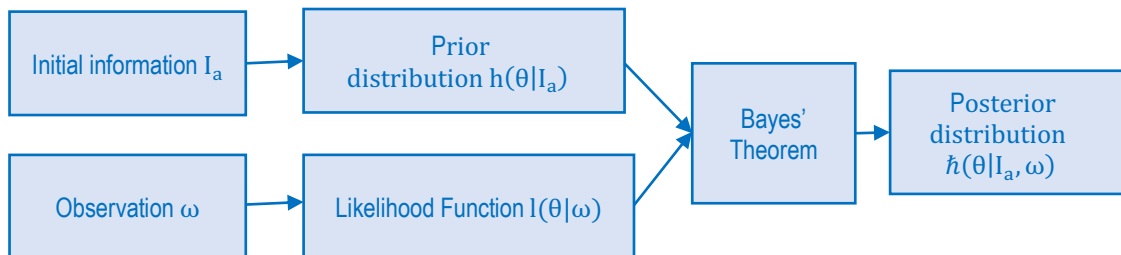
Source: (Zellner's, 1996).

It is important to notice that as sample information accumulates, it prevails in the posterior distribution, increasingly concentrated around the parameter's actual value. If two researchers had different prior distributions (due perhaps to different initial information), their posterior distributions would converge.

Continue discussing Stress-Strength Model and return to the Risk Metric in the form of (2.3). Now, it is presented as a function of $\theta \in \Theta$:

$$R(\theta) = \int_{U(x)<0} f(x|\theta)dx \qquad (4.2)$$

The problem is to find the prior and posterior estimators for $R(\theta)$. These estimators depend on the choice of the loss function. As mentioned above, the squared-error loss function is used in the most practical application, leading to the estimator as an expected value. Thus, given the prior $h(\theta|I_a)$ and posterior $\hbar(\theta|I_a, \omega)$ probability distribution, we can obtain Prior Estimator of Risk metric:

$$R_P = \int_\Theta R(\theta)\, h(\theta|I_a)d\theta, \qquad (4.3)$$

as well as the Posterior one:

$$R_{PS} = \int_\Theta R(\theta)\, \hbar(\theta|I_a, \omega)d\theta. \qquad (4.4)$$

To illustrate this technique, return to the example of section 2 when estimating the risk of not achieving the required margin M. It is assumed that all random variables, namely, price, product volume, per unit variable costs, and total fixed costs, are Gaussian distributed with given amounts for all their parameters (means values, standard deviations, and correlation coefficients). We assume they are uncertain, and their prior distributions are given. To simplify the following discussion, we restrict the case by one product in the company portfolio and assume that the mean values are uncertain and the standard deviations are known precisely.

According to these assumptions, $\theta = (\mu_p, \mu_v, \mu_Q, \mu_F)$, where we use μ_k instead m_k to stress that the mean values are random. Now the considered risk metric is presented as

$$R(\theta) = R(\mu_p, \mu_v, \mu_Q, \mu_F) = Pr.\{U(\boldsymbol{X}) < 0\} = \Phi\left(\frac{\mu_U}{\sigma_U}\right), \qquad (4.5)$$

where:

$$\mu_U = (\mu_p - \mu_v) \cdot \mu_Q - \mu_F, \qquad (4.6)$$

$$\sigma_U^2 = \mu_Q^2(\sigma_p^2 + \sigma_v^2) + (\mu_p - \mu_v)^2 \sigma_Q^2 + \sigma_F^2. \qquad (4.7)$$

The general procedure of computation of the prior and posterior estimates can be presented by the following steps:

- Step 1. Based on initial information available or subjective judgments, assign the prior probability distribution of $\theta = (\mu_p, \mu_v, \mu_Q, \mu_F)$.
- Step 2. Combined (4.6) and (4.7) with (4.5) and then with (4.3), calculate the Prior estimate of risk metric R_P.
- Step 3. Based on Bayes' formula (4.1), given a sample of observation ω, gain the posterior distribution $\hbar(\theta|I_a, \omega)$.
- Step 4. Combined (4.6) and (4.7) with (4.5) and then with (4.4), calculate the Posterior estimate of risk metric R_{PS}.

Apply this procedure to the example started in Section 2. We consider a single-product portfolio with the values presented in Table 4.1.

Table 4.1. Products' portfolio (all monetary numbers are in $1000)

Product #1	Mean Value	St. Dev.
Price per Unit	10,00	1,00
Variable Cost per Unit	5,50	0,60
Sale Volume	30,00	3,00
Total Fixed Costs	50,00	5,00

Using (2.6) and (2.8), we can calculate the Mean Value and Standard Deviation of the Capability Function U: $m_U = \$85.00K$, $\sigma_U = \$37.83K$. If the management requires EBIT value M $= \$70K$, using (2.9), we find the risk estimate as 0.35.

We can now proceed to obtain Bayesian risk estimates. In order to do this, we first need to set the prior distributions of mean values for per-unit price, per-unit variable cost, sale volume, and total fixed costs. We will assume that these parameters follow a uniform distribution in the interval $[\theta', \theta'']$. This refers to a situation where it is possible to determine the range boundaries in which the parameter exists. Still, favouring any particular points within that range is impossible. Table 4.2 presents prior information for the discussed case.

Table 4.2. Boundaries of the mean values

	μ'	μ''
μ_p	$8,000	$12,000
μ_Q	25	35
μ_v	$3,000	$8,000
μ_F	$35,000	$65,000

According to step 2, we must calculate the integral (4.3) in dimension four. Unfortunately, we cannot solve this problem using closed analytical methods for the chosen prior distributions. As a result, we must use a numerical solution. To do this, we employed Monte Carlo simulation, which involves sampling a large number of arguments for the integrable function and calculating the value of the function for each of these arguments. By summing up the calculated values of the function with high precision, we are able to determine the integral value. The accuracy of our results is directly proportional to the number of argument values we consider, denoted by N. The final equation for an approximate value of R_P is presented as follows:

$$R_P \cong \frac{\sum_{k=1}^{N} \prod_{i=1}^{4} h(\theta_i^k) R(\theta_i^k)}{\sum_{k=1}^{N} \prod_{i=1}^{4} h(\theta_i^k)}. \tag{4.8}$$

We used a sample size $N = 1000$ and finally obtained $R_P = 0.43$. It is quite understandable that the risk score turned out to be higher, given that assuming ambiguity in the model parameters increases the degree of uncertainty, which inevitably leads to a higher risk assessment.

To proceed to step 3, we must make observations and record the actual values of the primary parameters in the risk assessment model. These parameters included per-unit price, per-unit variable cost, sale volume, and total fixed costs and were denoted by ω. A sample of observed values, which are the monthly statistics of the primary parameters, is presented in Table 4.3

Table 4.3. Sample of primary variables (all monetary numbers are in $1000)

Month	Per-unit price	Sale volume	Per-unit variable cost	Total fixed costs
Jun	9.00	25	7.30	55.00
Feb	8.50	23	7.20	46.00
March	9.80	26	6.80	50.00
Apr	10.10	28	6.90	54.00
May	11.00	32	6.30	45.00
June	12.00	35	5.60	56.00
July	12.50	37	5.90	48.00
Aug	11.50	35	5.20	51.00
Sep	11.50	33	5.50	52.00

Month	Per-unit price	Sale volume	Per-unit variable cost	Total fixed costs
Oct	13.00	38	4.50	46.00
Nov	12.50	39	4.80	54.00
Dec	12.80	36	3.70	50.00

According to the assumption regarding the Gaussian probability distribution of the primary variables (4.1) can be rewritten as:

$$\hbar(\theta|I_a, \omega) = \beta \prod_{i=1}^{4} h(\theta_i) l(\theta; \omega) \tag{4.9}$$

where: the likelihood function can be expressed by the following equation:

$$l(\theta; \omega) = \prod_{k=1}^{n} \prod_{i=1}^{4} \varphi(x_i^{(k)}; \theta_i, \sigma_i) \tag{4.10}$$

and $\varphi(x; m, \sigma)$ is a Gaussian probability density function.

In equation (4.7), β represents the coefficient that yields a normalized posterior probability density function:

$$\beta = \left[\int_{\Theta} h(\theta) \, l(\theta; \omega) d\theta, \right]^{-1} . \tag{4.11}$$

The last fourth step allows us to obtain the posterior risk estimate. Here we will again have to use Monte Carlo simulation, for which we will apply formula (4.8), in which we will replace the prior distribution $h(\theta)$ with the posterior one, which is obtained using (4.9) – (4.11). The estimate obtained as a result of the calculations was $R_{PS} = 0.48$. The observations made increased the risk of achieving a given margin M = $70K$.

The fourth and final step enables us to estimate the posterior risk. Once again, we need to apply Monte Carlo simulation using formula (4.8), but this time we will substitute the prior distribution $h(\theta)$ with the posterior distribution obtained from (4.9) to (4.11). After the calculations, we obtained an estimate of $R_{PS} = 0.48$. Furthermore, the observations have augmented the risk of achieving a specified margin of margin M = $70K$.

An interesting pattern is observed if we compare the prior and the posterior estimates with increasing requirements regarding to the required margin *M*. The graph in Figure 4.1 shows that the posterior estimate has a higher growth rate as *M* increases. For moderate values of the required margin (from zero to $60K), the prior estimate is lower than the posterior estimate. However, when crossing the indifference point (around $60K), the posterior estimate is always higher. If the margin requirement increases significantly, both risk scores will eventually converge to one.

Figure. 4.1. Risk estimates vs margin *M*
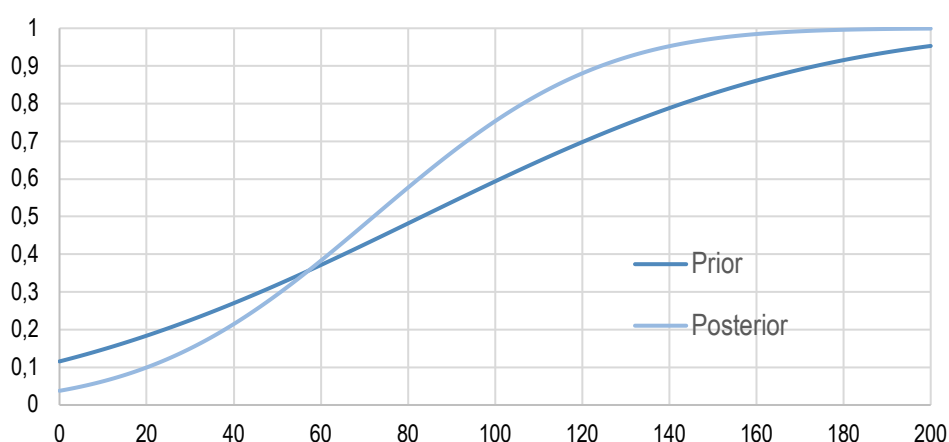
Conclusions

The considered task involved comparing two risk assessment methodologies based on the Stress-Strength model: classical and Bayesian. The Bayesian method considers the parameters of the system to be random and takes into account subsequent experiments in the design process. Bayesian estimators can be seen as a generalization of classical techniques. The classical assessment offers simplicity and quick risk assessments but relies on prior knowledge of the parameters of the model's variables. This information is not always available in practice.

The Bayesian approach removes this assumption and presents prior information about the variables less strictly and responsibly. However, the Bayesian approach allows decision-makers to refine prior representation as values of primary variables are obtained. This comes at the cost of more complicated computational schemes.

The proposed method is thoroughly illustrated using numerical examples, which enable a decision-maker to understand the general sequence of evaluation and apply it in specific practice.

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Credit Authorship Contribution Statement

The author performed all tasks involved in manuscript preparation, research, and writing.

Conflict of Interest Statement

The author declares that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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