A Note on Minimality in Dynare

Alessandro SACCAL
Independent Researcher, Italy
saccal.alessandro@gmail.com

Article's history:

Received 20th of August, 2021; *Received in revised form* 28th of September, 2021; *Accepted* 14th of October, 2021; *Published* 25th of October, 2021.

Suggested citation:

Saccal, A. 2021. A Note on Minimality in Dynare. *Journal of Applied Economic Sciences*, Volume XVI, Fall, 3(73): 318–323.

Abstract:

Since January 2014 this note and a manuscript entailing it have shown that the syntactic implication "Minimal linear time invariant state space representations *if dynare*" is false, with consequences on the vector autoregression representations of the states in the outputs. In 2020 the *dynare* team materially adopted the remedy of reducing its representations to minimal ones, as this note and the manuscript entailing it had been suggesting. The interested *dynare* user must still manually reduce the representation to minimality.

Keywords: *dynare*; minimality; state space.

JEL Classification: C02; C32.

Introduction

In the main, *dynare* is a *CEPREMAP* (2011) dynamic stochastic general equilibrium model solver for *Matlab* or *Octave.*

According to Franchi and Paruolo (2014), Komunjer and Ng (2011) show that *dynare* delivers non-minimal linear time invariant state space representations; such is not (immediately) verifiable. In December 2013 *dynare* programmer Johannes Pfeifer communicated to this author on the *dynare* internet forum through private messages (see Appendix) that *dynare* delivers minimal linear time invariant state space representations; this author denied it. This author circulated manuscripts proving it since January 2014, from the *University of Rome "Tor Vergata"*, some of which were extended to the *dynare* series team for consideration and submitted to academic journals, unsuccessfully. In January 2016 this author mentioned his finding to *dynare* programmer Marco Ratto during an interview at the *Joint Research Centre* in Ispra. In March 2020 *dynare* programmer Johannes Pfeifer indicated on the *dynare* internet forum that *dynare* augments, no less than sometimes, its minimal linear time invariant state space representations to non-minimal ones. In January 2020 *dynare* programmer Willi Mutschler had written a code allowing *dynare* to deliver minimal linear time invariant state space representations and in May 2020 he linked to it on the *dynare* internet forum.

Until Mutschler's code, this note and the manuscript "Structural shocks empirical recovery under minimal linear state space systems" entailing it showed *dynare*'s failure to deliver minimal linear time invariant state space representations, which compromised the check for vector autoregression representations of the states in the outputs: they suggested the remedy eventually coded by Mutschler. This note presents its years long finding in a definitive fashion, logically and explicitly showing why Mutschler's code was necessary, responding to what an anonymous referee of *Economics Letters* had commented in October 2015 (see Appendix) to the manuscript "Structural shocks empirical recovery under minimal linear state space systems" entailing it.

The only remark today left to make is that *dynare* adopted the remedy materially, not formally, for computational reasons, so that the manual reduction to minimality be still required by the interested *dynare* user.

1. State Space

Let function $f : \mathbb{R}^n \to \mathbb{R}^n,\ \forall n \geq 1,$ give rise to the first order linear heterogeneous difference equation

$$x_t = Ax_{t-1} + Bu_t,\ \forall t \in \mathbb{Z},\ x_t \in \mathbb{R}^{n_x},\ u_t \in \mathbb{R}^{n_u},\ A \in \mathbb{R}^{n_x \times n_x} \text{ and } B \in \mathbb{R}^{n_x \times n_u}.$$

It is the transition equation of a linear time invariant state space representation in discrete time, in which x_t is a vector of states and u_t a vector of inputs. The transition equation is also called state equation; inputs are also called controls *(i.e.* shocks).

Let $M \in \mathbb{R}^{n_y \times n_x}$ give rise to:

$$Mx_t = MAx_{t-1} + MBu_t \longleftrightarrow y_t = Cx_{t-1} + Du_t,\ \forall y_t \in \mathbb{R}^{n_y},\ C \in \mathbb{R}^{n_y \times n_x}$$

and $D \in \mathbb{R}^{n_y \times n_u}$.

It is the measurement equation of a linear time invariant state space representation in discrete time, in which ***u*** u_t is a vector of outputs; ***M*** is called measurement or observation matrix. The measurement equation is also called observation equation; outputs are also called observables.

Assume that ***D*** be non-singular and thus square: $n_y = n_u$. Solve for u_t the measurement equation and plug it into the transition equation:

$$y_t = Cx_{t-1} + Du_t \longrightarrow u_t = D^{-1}\left(y_t - Cx_{t-1}\right) \longrightarrow x_t = Ax_{t-1} + BD^{-1}\left(y_t - Cx_{t-1}\right)$$

$$\longrightarrow x_t = \left(A - BD^{-1}C\right)x_{t-1} + BD^{-1}y_t = Fx_{t-1} + BD^{-1}y_t.$$

Notice that: $F \equiv A - BD^{-1}C.$

Let operator $L : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ give rise to $Lx_t = x_{t-1}$. Solve $x_t = Fx_{t-1} + BD^{-1}y_t$ backwards:

$$x_t = Fx_{t-1} + BD^{-1}y_t \longrightarrow (I - FL)\,x_t = BD^{-1}y_t \longrightarrow x_t = (I - FL)^{-1}BD^{-1}y_t$$

$$\longrightarrow x_t = \sum_{j=0}^{\infty} F^j L^j BD^{-1}y_t \longrightarrow x_t = \sum_{j=0}^{\infty} F^j BD^{-1}y_{t-j}.$$

Notice that $x_t = \sum_{j=0}^{\infty} F^j BD^{-1}y_{t-j}$ is causal *if and only if* ***F*** is stable, namely, ***F*** 's characteristic polynomial eigenvalues are less than one in modulus:

$|\lambda_{F(\lambda)}| < 1$ for $F(\lambda) = F - \lambda I$ in $det\left[F(\lambda)\right] = 0$, so that:

$$(I - FL)^{-1} = \sum_{j=0}^{\infty} F^j L^j \longleftrightarrow I = (I - FL)\sum_{j=0}^{\infty} F^j L^j.$$

Plug $x_t = \sum_{j=0}^{\infty} F^j BD^{-1}y_{t-j}$ into the measurement equation: $y_t = C\sum_{j=0}^{\infty} F^j BD^{-1}y_{t-j-1} + Du_t.$

2. Vector Auto Regression and Minimality

Fernández-Villaverde *et al.* (2007) prove that $y_t = C\sum_{j=0}^{\infty} F^j BD^{-1}y_{t-j-1} + Du_t$ is a vector autoregression of infinite order $VAR(\infty)$ *if* F is stable: there exists a VAR of x_t in y_t. Ravenna (2007) and

Franchi and Vidotto (2013) prove that $y_t = C\sum_{j=0}^{k} F^j BD^{-1} y_{t-j-1} + Du_t$ is a vector autoregression of finite order $VAR(k)\ \text{for}\ k < \infty$ *if* F is nilpotent, namely, ***F's*** characteristic polynomial eigenvalues are zero: $\lambda_{F(\lambda)} = 0.$

Franchi (2013) and Franchi and Paruolo (2014) remark that ***F's*** stability is sufficient but unnecessary for a VAR of x_t in y_t, because a stable, minimal ***F*** could give rise to one: in minimal linear time invariant state space representations the impulse response functions of the transition equation and the coefficients of the VAR representation of x_t in y_t, are invariant (see Franchi (2013)). Thus, $F = F_m$'s stability is both sufficient and necessary for a VAR x_t in y_t,

A minimal linear time invariant state space representation is computed in three steps. *Step A.* Construct controllability matrix $\mathcal{C} = \left[\cdots\ A^{n_x-1}B\right]$ and observability matrix $\mathcal{O} = \left[\cdots\ CA^{n_x-1}\right]^\top$.
Step B. If rank $n_x = r_{\mathcal{C}}$, the representation is controllable: $\bar{x}_{ct},\ \bar{A}_c,\ \bar{B}_c,\ \bar{C}_c,\ \bar{\mathcal{C}}_c$ and $\mathcal{T} = \left[\mathcal{C}_{r\mathcal{C}}\ v_{n_x - r\mathcal{C}}\right]$ $\bar{\mathcal{O}}_c;$ go to Step C. If $n_x > r_{\mathcal{C}},$ construct similarity transformation matrix such that:

$$\bar{x}_{c\bar{c}t} = \mathcal{T}^{-1}x_t,\ \bar{A}_{c\bar{c}} = \mathcal{T}^{-1}A\mathcal{T},\ \bar{B}_{c\bar{c}} = \mathcal{T}^{-1}B,\ \bar{C}_{c\bar{c}} = C\mathcal{T},\ \bar{\mathcal{C}}_{c\bar{c}} = \mathcal{T}^{-1}\mathcal{C},\ \bar{\mathcal{O}}_{c\bar{c}} = \mathcal{O}\mathcal{T}.$$

The representation is controllable in the first $r_{\mathcal{C}}$ states: $\bar{x}_{ct},\ \bar{A}_c,\ \bar{B}_c,\ \bar{C}_c,\ \bar{\mathcal{C}}_c$ and $\bar{\mathcal{O}}_c;$ go to Step C. *Step C.* If $n_{\bar{x}_c} = r_{\bar{\mathcal{O}}_c}$, the representation is controllable and observable (*i.e.* minimal): $\bar{x}_{cot} = x_{mt},\ \bar{A}_{co} = A_m,\ \bar{B}_{co} = B_m,\ \bar{C}_{co} = C_m,\ \bar{\mathcal{C}}_{co} = \mathcal{C}_m$ and $\bar{\mathcal{O}}_{co} = \mathcal{O}_m.$ If If $n_{\bar{x}_c} > r_{\bar{\mathcal{O}}_c},$ construct similarity transformation matrix $\mathcal{T} = \left[\bar{\mathcal{O}}_{cr_{\bar{\mathcal{O}}_c}}\ v_{n_{x_c} - r_{\bar{\mathcal{O}}_c}}\right]^\top$ such that

$$\bar{x}_{co\bar{o}t} = \mathcal{T}^{-1}\bar{x}_{ct},\ \bar{A}_{co\bar{o}} = \mathcal{T}^{-1}\bar{A}_c\mathcal{T},\ \bar{B}_{co\bar{o}} = \mathcal{T}^{-1}\bar{B}_c,\ \bar{C}_{co\bar{o}} = \bar{C}_c\mathcal{T},$$

$$\bar{\mathcal{C}}_{co\bar{o}} = \mathcal{T}^{-1}\bar{\mathcal{C}}_c,\ \bar{\mathcal{O}}_{co\bar{o}} = \bar{\mathcal{O}}_c\mathcal{T}.$$

The representation is controllable and observable (*i.e.* minimal) in the first $r_{\bar{\mathcal{O}}_c}$ states: $\bar{x}_{cot} = x_{mt},\ \bar{A}_{co} = A_m,\ \bar{B}_{co} = B_m,\ \bar{C}_{co} = C_m,\ \bar{\mathcal{C}}_{co} = \mathcal{C}_m$ and $\bar{\mathcal{O}}_{co} = \mathcal{O}_m.$ Notice that reducing the representation to observability before controllability leaves the algorithm unvaried: the order is a matter of (synthetic) expedience.

Consider the case of $C = 0$. Notice that $n_x > r_{\mathcal{O}} = 0,$ so that $x_{mt} = F_m = |\lambda_{F_m(\lambda)}| = 0;$ specifically, $x_{mt} = A_m x_{mt-1} + B_m u_t \longleftrightarrow 0 = 0$ and $y_t = Du_t.$ Yet, ***F's*** stability would be unnecessary for a VAR of x_t in $y_t,$ because $F = A - BD^{-1}C = A$ and $|\lambda_{A(\lambda)}| \gtreqless 1.$

3. Dynare State Space

Following the Blanchard and Kahn (1980) solution algorithm of linear rational expectations models, *dynare* gives rise to the unique and stable solution $X_t = [a\ c]^\top p_{t-1} + [b\ d]^\top n_t,\ AX_t = [p_t\ n_t]^\top,$ in which $p_t \in \mathbb{R}^{n_p}$ is a vector of predetermined states (*i.e.* appearing only at ***t*** and ***t - 1***) and $n_t \in \mathbb{R}_{n_n}$ is a vector of non-predetermined states. In economics they are also respectively called backward and forward looking states; in economics they are also respectively called states and controls, but in control theory controls are what economics calls shocks.

Dynare constructs the transition equation by selecting the first $\boldsymbol{n_p}$ rows of $\boldsymbol{X_t}$ and the measurement equation by selecting the measurable rows thereof (*i.e.* the measurable rows of the first $\boldsymbol{n_p}$ rows of $\boldsymbol{X_t}$): $x_t = p_t$ and $y_t = Mx_t = Mp_t.$

If p_t is fully measured then $F = 0$, because $M = I$, and *dynare* tautologically gives rise to a minimal linear time invariant state space representation. All cases are however such that the syntactic implication "Minimal linear time invariant state space representations *if dynare*" is false, as proven by the permanent income model *counterexample* below:

$D \not\longrightarrow MR$, since $\exists x \in U$ such that $Dx \wedge \neg MRx$, in which $D \equiv$ *dynare*, $MR \equiv$

Minimal representation, $x \equiv$ counterexample and $U \equiv$ universe (*i.e.* domain of discourse).

4. Minimality in Dynare

Consider the permanent income model: $c_t = c_{t-1} + \sigma_w \left(1 - r^{-1}\right) w_t$ (consumption); $y_{pt} = \sigma_w w_t$ (income); $s_t = y_{pt} - c_t$ (savings). $w_t \sim \mathcal{N}\left(0,\ \sigma^2\right)$ is an income shock modeled as a white noise; $r,\ \sigma_w \in \mathbb{R}$ are structural parameters, namely, the real interest rate and income shock variance; let $r = 1.2$ and $\sigma_w = 1$.

Let y_{pt} be measurable and construct the linear time invariant state space representation: $x_t = \left[c_t\ y_{pt}\ s_t\right]^\top;\ y_t = y_{pt};\ A = \left[(1\ 0\ 0)\ \ (0\ 0\ 0)\ \ (-1\ 0\ 0)\right]^\top;$ $B = \left[\sigma_w \left(1 - r^{-1}\right)\ \sigma_w\ \sigma_w r^{-1}\right]^\top;\ C = [0\ 0\ 0];\ D = \sigma_w.$ Compute $F,\ F(\lambda)$ and $\left|\lambda_{F(\lambda)}\right|:\ F = A;\ A(\lambda) = \left[(1 - \lambda\ 0\ 0)\ \ (0\ \ -\lambda\ 0)\ \ (-1\ 0\ \ -\lambda)\right]^\top;\ \left|\lambda_{A(\lambda)}\right| = 0_2,\ 1.$

There does not exist a $VAR(\infty)$ of x_t in y_t. Construct $\mathcal{O}$ and record $r_{\mathcal{O}}:\ \mathcal{O} = \left[(0\ 0\ 0)\ \ (0\ 0\ 0)\ \ (0\ 0\ 0)\right]^\top;\ r_{\mathcal{O}} = 0.$

Then $x_{mt} = F_m = \left|\lambda_{F_m(\lambda)}\right| = 0;$ specifically, $x_{mt} = A_m x_{mt-1} + B_m u_t \longleftrightarrow 0 = 0$ and $y_t = Du_t \longleftrightarrow y_{pt} = \sigma_w w_t.$

There exists a $VAR(k)$ of x_t in y_t. In *dynare* formally $x_t = c_t,\ A = 1,\ B = \sigma_w\left(1 - r^{-1}\right),\ C = 0,\ D = \sigma_w$ and $F = A = \left|\lambda_{A(\lambda)}\right| = 1:$ there does not exist a $VAR(\infty)$ of x_t in y_t.

Notice that $n_x > \mathcal{O} = r_{\mathcal{O}} = 0$, so that $x_t \neq x_{mt} = 0$ and $F \neq F_m = \left|\lambda_{F_m(\lambda)}\right| = 0:$ there exists a $VAR(k)$ of x_t in y_t.

To execute this place *ABCD_test.m* inside the *dynare* "Matlab" folder; run *FV_et_al_2007_ABCD.mod* with y_{pt} as the only measurable variable; construct $\mathcal{O}$ and record $r_{\mathcal{O}}$ by running *Obs=[C]* and *ro=rank(Obs)*; compute $A_m,\ B_m,\ C_m$ and $D_m = D$ by running *[Am, Bm, Cm, Dm] = minreal[A, B, C, D]*; compute F_m and $\lambda_{F_m(\lambda)}$ by running *Fm=Am-Bm*inv(D)*Cm* and *eig(Fm)*.

Dynare materially adopts minimal linear time invariant state space representations through Mutschler's code. Yea, in November 2020, posterior to the registration of this note on the MPRA, Pfeifer amended both *ABCD_test.m* and *FV_et_al_2007_ABCD.mod*, pertinently conveying that the check would be necessary and sufficient (if and) only if conducted on a minimal state space; in October 2021 he further amended *ABCD_test.m* to the same effect and presented *FV_et_al_2007_ABCD_minreal.mod,* wherein the above application of the permanent income model is realized (see Appendix as well).

Conclusion

This note's conclusion prescribes the subjective adoption of Mutschler's code or the like to the end of computing minimal linear time invariant state space representations in *dynare*.

References

[1] Adjemian, S., Bastani, H., Juillard, M., Karamé, F., Maih, J., Mihoubi, F., Mutschler, W., Perendia, G., Pfeifer, J., Ratto, M. and Villemot, S. 2011. Dynare: Reference Manual, Version 4, *Dynare Working Papers, 1, CEPREMAP.* Available at: https://www.dynare.org/manual.pdf

[2] Blanchard, O. and Kahn, C. 1980. The Solution of Linear Difference Models Under Rational Expectations, *Econometrica,* 48(5): 1305-1311. DOI: https://doi.org/10.2307/1912186

[3] Fernández-Villaverde, J., Rubio-Ramírez, J., Sargent, T. and Watson, M. 2007. ABCs (and Ds) of Understanding VARs, *American Economic Review,* 97(3): 1021-1026. DOI: https://doi.org/10.1257/aer.97.3.1021

[4] Franchi, M. 2013. Comment on: Ravenna, F., 2007. Vector Autoregressions and Reduced Form Representations of DSGE Models. *Journal of Monetary Economics*, 54: 2048-2064, *Dipartimento di Scienze Statistiche Empirical economics and Econometrics Working Papers Series.* Available at: https://www.dss.uniroma1.it/RePec/sas/wpaper/20132_Franchi.pdf

[5] Franchi, M. and Paruolo, P. 2014. Minimality of State Space Solutions of DSGE Models and Existence Conditions for their VAR Representation, *Computational Economics,* 46(4): 613-626. DOI: https://doi.org/10.1007/s10614-014-9465-4

[6] Franchi, M. and Vidotto, A. 2013. A Check for Finite Order VAR Representations of DSGE Models, *Economics Letters,* 120(1): 100-103. DOI: https://doi.org/10.1016/j.econlet.2013.04.013

[7] Komunjer, I. and Ng, S. 2011. Dynamic Identification of Dynamic Stochastic General Equilibrium Models, *Econometrica,* 79(6): 1995-2032. Available at: https://www.jstor.org/stable/41336541

[8] Ravenna, F. 2007. Vector Autoregressions and Reduced Form Representations of DSGE Models, *Journal of Monetary Economics,* 54(7): 2048 - 2064. DOI: https://doi.org/10.1016/j.jmoneco.2006.09.002

APPENDIX

Extract of private messages of dynare *programmer Johannes Pfeifer to this author in* dynare *Internet forum (December 11-12, 2013).*

- FV et al.'s state space only contains predetermined variables. Their CODE: SELECT ALL
x_t=Ax_t-1+Be_t

is exactly equal to the first n_x rows of CODE: SELECT ALL
[x_t; \tilde y_t]=[A;C]*x_t +[B;D]*e_t

The state-space i.e. the x_t is still minimal.

Regarding
CODE: SELECT ALL
H[x_t; \tilde y_t]=H[A;C]*x_t +H[B;D]*e_t

that is exactly what the ABCD_test function does by selecting only the rows belonging to states and observables.
However, H is a selector matrix only for the rows, not the columns with the latter still containing only the states.
Thus, the ABCD test in my function and in FV is necessary and sufficient (given that D is invertible)

Economics Letters comment to manuscript "Structural shocks empirical recovery under minimal linear state space systems" (October 13, 2015).

Reviewer #1:

The author(s) claim(s) that Dynare delivers non-minimal state-space system representations. Despites its contribution to the literature, unfortunately, the main result is somehow lost in the paper, because the author(s) devote several sections for replication of still existing results instead of being more focused on what is really new. Further, I think the claim is not sufficiently proved by the author. The author somewhat devotes a small section in claim 3.10 but is puzzling to me whether this claim is proved on the basis of an explicit model, like the PI model, without an explicit numerical application, which compares the Dynare result with the author's result. It is not sufficient to simply attach the codes, but to apply them. This would indeed highlight the result more clearly. In this light, the paper should be completely re-written, and hence, spend more space for a proper discussion of the robustness of the obtained results.

Extract of *DSGE_mod/README.md* *by Johannes Pfeifer as at October 25, 2021.*

FV_et_al_2007

Provides codes for the ABCD-test of Fernandez-Villaverde, Rubio-Ramirez, Sargent, and Watson (2007), "ABCs (and Ds) of Understanding VARs", American Economic Review, 97(3), pp. 1021-1026

Includes the `ABCD_test.m`. Note that it tests only a sufficient condition, not a necessary one, if the minimal state space is not computed. For details, see e.g. Komunjer/Ng (2011): "Dynamic Identification of Dynamic Stochastic General Equilibrium Models", Econometrica, 79(6), 1995–2032.

FV_et_al_2007.mod

Replicates the ABCD-test for the example of the permanent income model provided in Fernandez-Villaverde, Rubio-Ramirez, Sargent, and Watson (2007), "ABCs (and Ds) of Understanding VARs", American Economic Review, 97(3), pp. 1021-1026

FV_et_al_2007_ABCD_minreal.mod

Shows how to compute the minimal state space using Matlab's Control toolbox for the example of Saccal, Alessandro (2020): "A note on minimality in Dynare", available at https://mpra.ub.uni-muenchen.de/103656/1/MPRA_paper_103656.pdf.